Direct Reflections

To Horst Herrlich for his sixtieth birthday – with affection and gratitude

G. C. L. BRÜMMER
Department of Mathematics and Applied Mathematics, University of Cape Town, Rondebosch 7701, South Africa, e-mail: gclb@maths.uct.ac.za

E. GIULI
Dipartimento di Matematica Para ed Applicata, Università degli Studi di L’Aquila, via Vetoio, loc. Coppito, 67100 L’Aquila, Italia, e-mail: Giuli@aquila.infn.it

D. B. HOLGATE
Departement Wiskunde, Universiteit van Stellenbosch, Stellenbosch 7600, South Africa, e-mail: DH2@land.sun.ac.za

(Received: 26 February 1998; accepted: 18 January 1999)

Abstract. A pointed endofunctor (and in particular a reflector) \((R, r)\) in a category \(X\) is direct iff for each morphism \(f: X \to Y\) the pullback of \(Rf\) against \(r_Y\) exists and the unique fill-in morphism \(u\) from \(X\) to the pullback is such that \(Ru\) is an isomorphism. (This is close to the concept of a simple reflector introduced by Cassidy, Hébert and Kelly in 1985.) We give sufficient conditions for directness, and for directness to imply reflectivity. We also relate directness to perfect morphisms, and we give several examples and counterexamples in general topology.


Key words: direct reflection, perfect morphism, Samuel compactification, topological bicompletion.

Introduction

In their paper [8] Cassidy, Hébert and Kelly considered, among many other things, a case where a reflector \(R\) with unit \(r: 1 \to R\) in a category \(X\) gives rise to a morphism factorisation structure \((\Sigma_R, \Sigma_R^+)\) on \(X\) in a particularly simple way. For any morphism \(f: X \to Y\) they took the pullback \((P, (p_f, q_f))\) of \(Rf\) against \(r_Y\), assuming for this purpose \(X\) to be finitely complete. There is the unique morphism \(u_f: X \to P\) with \(f = p_f u_f\) and \(r_X = q_f u_f\). If for every morphism \(f, u_f \in \Sigma_R\) (i.e. \(Ru_f\) is an iso), then \(R\) is called a simple reflector and the desired factorisation is indeed \(f = p_f u_f\).

In the above paper and in some recent work (e.g. [5, 14, 15]) the concept of a simple reflector has had applications in particular to certain variants of the concept of perfect morphism. We have found it helpful to free the definition of simple reflector from the restriction that the category be finitely complete. Accordingly
we say that the reflector $R$ is *direct* if for each $f: X \to Y$ the pullback of $Rf$ against $r_Y$ exists and $u_f \in \Sigma_R$. Much of the theory persists without $R$ being a reflector, so we consider arbitrary pointed endofunctors $(R, r)$.

In this paper we (1) give sufficient conditions for a direct $(R, r)$ to be a reflector; (2) relate directness to factorisations and perfect morphisms; (3) relate perfect morphisms to products; (4) give elementary, ‘practical’ sufficient conditions for $(R, r)$ to be direct; (5) give a range of examples and counterexamples involving directness in general topology.

It should be noted that a great deal of literature has been devoted to categorical generalisations of perfect mappings, starting with Herrlich’s papers [10, 11]. A quick guide to such literature up to the present can be found in [7] and [15]. In the present paper a morphism $f: X \to Y$ will be called $R$-perfect if $f$ is a pullback of $Rf$ against $r_Y$. This notion coincides with the Henriksen–Isbell characterisation of perfect mappings [12] in case $(R, r)$ is the Stone–Čech compactification in $\text{Tych}$, but deviates from it when, for instance, $(R, r)$ is the Samuel compactification in the $T_0$-uniform spaces. Our Example 5.6 pinpoints the difference from approaches to perfection such as those of [7] and [9].

Categorical notation and terminology is that of [1].

1. Direct Endofunctors

Throughout this paper the pair $(R, r)$ denotes a pointed endofunctor on a category $X$, i.e., a functor $R: X \to X$ together with a natural transformation $r: 1 \to R$ [17]. The pointed endofunctor is said to be *idempotent* if for every object $X$ in $X$ the morphism $r_{RX}: RX \to R^2X$ is an isomorphism. It is said to be *well-pointed* if for any object $X$ in $X$, $Rr_X = r_{RX}$ (i.e., the natural transformations $rR = Rr$).

We will denote by $\Sigma_R$ the collection of all $X$-morphisms $f$ for which $Rf$ is an isomorphism, and by $\text{Fix}(R, r)$ the full subcategory of $X$ of those $X$-objects $X$ for which $r_X: X \to RX$ is an isomorphism.

**DEFINITION 1.1.** An endofunctor $(R, r)$ is said to be direct if for any $f: X \to Y$ the pullback $(P, (p_f, q_f))$ of $Rf$ along $r_Y$ exists and the induced morphism $u_f: X \to P$ is in $\Sigma_R$ (see Diagram (*) below).

We will devote most of our attention to direct reflections. Direct endofunctors that are not reflections do exist, but we will see that under fairly mild conditions directness implies reflectivity.

\[ \begin{align*}
X & \xleftarrow{r_X} RX & \xrightarrow{Rf} & RY \\
& \quad \downarrow{u_f} & \quad \downarrow{Rf} \\
Y & \xleftarrow{r_Y} & P & \xrightarrow{p_f} \quad Y \\
& \quad \downarrow{q_f} & \quad \downarrow{r_Y} \\
& \quad \downarrow{p_f} & \quad \downarrow{r_Y} \\
& \quad \downarrow{q_f} & \quad \downarrow{r_Y} \\
\end{align*} \]

(*)
PROPOSITION 1.2. A pointed endofunctor \((R, r)\) is a reflection iff \((R, r)\) is well-pointed and idempotent.

Proof. The forward implication is clear. For the reverse implication, first note that \(RX \in \text{Fix}(R, r)\) for any \(X\). Let \(F \in \text{Fix}(R, r)\), and consider a morphism \(f: X \to F\). We have \(f^* := r_F^{-1}RF\) an extension of \(f\) through \(r_X\). If \(g\) were another such extension, then \(r_Fg = RG \circ r_RX = Rg \circ Rr_X = R(gr_X) = RF\) so \(g = f^*\).

PROPOSITION 1.3. Let \(X\) have a terminal object. If \((R, r)\) is direct and well-pointed it is idempotent and hence a reflection.

Proof. Let \(T\) be the terminal object in \(X\), and \(t_{RT}: RT \to T\) the unique terminal morphism.

Since \(t_{RT}t_T = 1_T\) and \((R, r)\) is well-pointed, \(1_{RT} = Rr_{RT} \circ r_T = R(t_{RT} \circ r_T) = r_Tt_{RT}\). Hence \(r_T\) is an isomorphism. Now for any \(X \in \text{Ob}X\) we can perform the construction below given by the directness of \((R, r)\).

Since \(r_T\) is an isomorphism, so is \(q\). Since \((R, r)\) is direct, \(r_X = qu \in \Sigma_R\) so that \(r_RX = Rr_X\) is an isomorphism and \((R, r)\) is idempotent. By the result above, it follows that \((R, r)\) is in fact a reflection. \(\square\)

PROPOSITION 1.4. A direct and idempotent \((R, r)\) will be a reflection if any of the following holds:

(a) \((R, r)\) is pointwise epimorphic.
(b) \(\Sigma_R\) is a class of epimorphisms.
(c) \((R, r)\) is pointwise monomorphic.

Proof. If we perform the construction below for \(X\) in \(X\), then \(r_RX\) is an isomorphism and hence its pullback \(q\) is an isomorphism and \(r_X \in \Sigma_R\).
This means that \( p \) is also an isomorphism and hence \( R_X \cong r_X \). Strict equality will follow if (a) \((R, r)\) is pointwise epimorphic (trivial), (b) \( \Sigma_R \subseteq \text{Epi} X \) (then in the diagram above \( p = q \)), or (c) \((R, r)\) is pointwise monomorphic (for such \((R, r)\) \( \Sigma_R \subseteq \text{Epi} X \)).

REMARK 1.5. As can be seen from the above propositions, if we are to give an example of a direct but nonreflective endofunctor then it is unlikely to be well-behaved. A trivial example of such an endofunctor is \((R, r)\) defined on \( \text{Set} \), where for a set \( X, R_X = \{0, 1\} \) with \( r_X(x) = 0 \) for every \( x \in X \) and \( r_X = \emptyset \). For \( f: X \to Y, Rf = \text{id}_{\{0, 1\}} \). Clearly this \((R, r)\) is neither well-pointed nor idempotent, it is however trivially direct since \( \text{Set} \) has pullbacks and \( \Sigma_R = \text{Mor}(\text{Set}) \).

2. Directness and Factorisation Theory

As mentioned in the introduction, the concept of directness is useful in the study of perfect morphisms. A morphism \( f \) is called \( R \)-perfect if in the diagram (**) in 1.1, \( u_f \) is an isomorphism. Such morphisms were studied in [13, 14, 15] together with several other categorical versions of the concept of perfect mapping.

The results below were utilised to some extent in [13, 14, 15]. The following summary presents some improvements, and collates them afresh.

**PROPOSITION 2.1.** If every \( X \)-morphism is \((\Sigma_R, R \text{-perfect})\) factorisable, then \((R, r)\) is direct.

**Proof.** This is proved in the reverse implication of [14, Proposition 16]. (The assumption of idempotence made there is not used in that part of the proof.)

**PROPOSITION 2.2.** If \((R, r)\) is idempotent or \( \Sigma_R \subseteq \text{Epi} X \) then if \((R, r)\) is direct,
every \( X \)-morphism is \((\Sigma_R, R \text{-perfect})\) factorisable.

**Proof.** Since \((R, r)\) is direct we can form the pullback \((P, (p_f, q_f))\) of \((Rf, ry)\).

Then \( Ru_f \circ q_f u_f = Ru_f \circ r_X = r_P u_f \) from which, in case \( \Sigma_R \subseteq \text{Epi} X \), we conclude that \( Ru_f \circ q_f = r_P \). The same conclusion is reached, in case \((R, r)\) is idempotent, in the proof of [14, Proposition 16]. In either case \( Ru_f \) is an isomorphism, so the square \( r_Y p_f = Rp_f \circ r_P \) is a pullback giving that \( p_f u_f \) is a \((\Sigma_R, R \text{-perfect})\) factorisation of \( f \).

The previous two propositions now combine to give the theorem below, which reinforces our contention that directness as against simplicity is the correct notion.
to consider. The existence of pullbacks in $X$ is not needed for the characterisation given here. The necessary pullbacks are inherent in directness on the one hand, and the factorisation system on the other.

**THEOREM 2.3.** If $(R, r)$ is idempotent or $\Sigma R \subseteq \text{Epi} X$ then: $(R, r)$ is direct iff $(\Sigma R, R$-perfect) is a factorisation structure for morphisms in $X$.

Proof. This is Corollary 17 of [14] with the additional option of $\Sigma R \subseteq \text{Epi} X$ which is available due to Proposition 2.2. \hfill $\square$

We refer to [8] for the notation $A^\downarrow$ for a morphism class $A$.

**COROLLARY 2.4.** If $(R, r)$ is idempotent or $\Sigma R \subseteq \text{Epi} X$, then if $(R, r)$ is direct it follows that $R$-perfect $= \Sigma^\downarrow_1$.

**REMARKS 2.5.** This corollary tells us that for such $(R, r)$ the $R$-perfect morphisms inherit the properties of $\Sigma^\downarrow_1$. For instance they are closed under arbitrary products and pullbacks.

Clearly the results of this section hold for reflectors, $R$. From Proposition 1.3 we see that if $X$ has a terminal object, they hold for well-pointed $(R, r)$. Also if $(R, r)$ is pointwise monomorphic they hold true, as for such $(R, r)$, $\Sigma R \subseteq \text{Epi} X$.

The strength of the link between directness and equality of $R$-perfect and $\Sigma^\downarrow_1$ morphisms was first revealed by the following theorem.

**THEOREM 2.6** [8]. In a category with pullbacks, a reflector $R$ is direct iff $R$-perfect $= \Sigma^\downarrow_1$.

### 3. Perfect Morphisms and Products

**LEMMA 3.1.** For any $(R, r)$, if $\Pi X_i$ is a product with $X_i \in \text{Fix}(R, r)$ for all $i \in I$, $i \neq j$, then the projection $\pi_j: \Pi X_i \to X_j \in \Sigma^\downarrow_1$.

Proof. Let $\Pi X_i$ be a product in $X$ and $j \in I$ such that $X_j \in \text{Fix}(R, r)$ for every $i \neq j$. Take a commutative square $\pi_j u = vh$ where $h \in \Sigma R$.

\[
\begin{array}{cccccc}
\Pi X_i & \xrightarrow{\pi_j} & X_j \\
\downarrow r_{\pi_j} & & & \downarrow & \\
RX_i & \xrightarrow{r_{X_j}} & X_i
\end{array}
\]

Let $d: B \to \Pi X_i$ be the unique morphism given by the universal property of the product for which $\pi_j d = v$ and $\pi_i d = r^{-1}_{X_i} R \pi_i \circ Ru \circ (Rh)^{-1} r_B$ for $i \in I$, $i \neq j$. 

Then for all \( i \in I \), \( \pi_i d h = \pi_i u \) so \( d \) is a diagonal for the commuting square. That it is unique in this role follows since if \( d^* \) is another diagonal then \( \pi_i d^* = \pi_i d \) and for \( i \neq j \), \( \pi_i d^* h = \pi_i u = r_{X_i}^{-1} R \pi_i \circ Ru \circ (Rh)^{-1} r_h = \pi_j d h \). Thus \( d = d^* \) because \( h \in \Sigma_R \) is Fix\((R, r)\)-cancellable (see Definition 4.3(a)), and \((\pi_i)_{i \in I}\) is a mono-source.

**PROPOSITION 3.2.** Let \( X \) have finite products, and \((R, r)\) be a reflector in \( X \) with \( R = \text{Fix}(R, r) \). For an \( X \)-object \( X \), the following are equivalent.

(a) \( X \in R \).
(b) For any \( X \)-object \( Y \), the projection \( \pi_Y: X \times Y \rightarrow Y \) is in \( \Sigma_R^1 \).
(c) The projection \( \pi_{RX}: X \times RX \rightarrow RX \) is in \( \Sigma_R^1 \).
(d) The unique map from \( X \) to the terminal object (empty product) is in \( \Sigma_R^1 \).

**Proof.**
(a) \( \Rightarrow \) (b) Above lemma.
(b) \( \Rightarrow \) (c) and (b) \( \Rightarrow \) (d) Trivial.
(c) \( \Rightarrow \) (a) Consider the commutative square \( 1_{RX}r_X = \pi_{RX} h \), where \( h: X \rightarrow X \times RX \) is the unique morphism such that \( \pi_X h = 1_X \) and \( \pi_{RX} h = r_X \). Since \( r_X \in \Sigma_R \), there is a unique diagonal \( d \) with \( d r_X = h \) and \( \pi_{RX} d = 1_{RX} \). Then \( \pi_X d r_X = \pi_X h = 1_X \) so \( r_X \) is a section and hence an isomorphism.
(d) \( \Rightarrow \) (a) Looking at the commutative square \( t_{RX}r_X = t_X 1_X \) with \( t_X \in \Sigma_R^1 \) and \( r_X \in \Sigma_R \) we see that \( r_X \) is a section and hence that \( X \in R \).

4. Sufficient Criteria for the Directness of a Pointed Endofunctor

The following two results are the most notable ones in the literature that give sufficient conditions for the directness of a reflection.

**THEOREM 4.1** [8]. Let \( X \) be finitely complete and \((R, r)\) a reflection in \( X \). Then (a) if (b) and each statement implies its successor.

(a) \( R \) is a localisation.
(b) \( \Sigma_R \) is closed under pullbacks.
(c) The pullback of \( r_X: X \rightarrow RX \) along any \( X \)-morphism is in \( \Sigma_R \).
(d) \( \Sigma_R \) is closed under pullbacks along any \( \Sigma_R^1 \) morphism.
(e) \( R \) is direct.

**PROPOSITION 4.2** [8]. If \( X \) has pullbacks, and these are preserved by a reflector \( R \), then \( R \) is direct.

Our intention is to find criteria that enable the ‘working topologist’ to decide whether a given pointed endofunctor or reflector is direct. Knowing that \((R, r)\) is direct then opens up many results about perfect maps and the factorisation systems associated with them ([8, 14, 15]).
DEFINITION 4.3. For a pointed endofunctor \((R, r)\) we shall write \(R := \text{Fix}(R, r)\). We define the following classes of \(X\)-morphisms.

(a) \(\mathcal{D}(R) = \{\text{all } R\text{-cancellable morphisms}\}\), where \(f : X \to Y\) in \(X\) is \(R\text{-cancellable}\) if for any \(s, t : Y \to A, A \in R, sf = tf\) implies \(s = t\).

(b) \(\mathcal{M}(R) = \{\text{all } R\text{-extendable morphisms}\}\), where \(f : X \to Y\) is \(R\text{-extendable}\) if any \(g : X \to A\) with codomain \(A \in \text{Ob } R\) extends through \(f\).

LEMMA 4.4. If \((R, r)\) is a pointed endofunctor and \(R = \text{Fix}(R, r)\), then \(\Sigma_R \subseteq \mathcal{D}(R) \cap \mathcal{M}(R)\). If also \((R, r)\) is a reflector, then \(\Sigma_R = \mathcal{D}(R) \cap \mathcal{M}(R)\).

Proof. The proof of equality for the reflective case in [5] also gives the inclusion in the general case.

PROPOSITION 4.5. Suppose that \(X\) has pullbacks of type \((\ast)\), and that there exist \(X\)-morphism classes \(\mathcal{E}\) and \(\mathcal{S}\) such that:

(a) For each \(X \in \text{Ob } X\), \(r_X \in \Sigma_R \cap \mathcal{E}\);
(b) For all pullbacks of type \((\ast)\), the morphism \(q_f \in \mathcal{S}\);
(c) For any \(X\)-morphisms \(e\) and \(g, eg \in \mathcal{E} \Rightarrow e \in \mathcal{E}\);
(d) \(\mathcal{E} \cap \mathcal{S} \subseteq \Sigma_R\).

Then \((R, r)\) is direct.

Proof. Since \(q_f u_f = r_X\), by (a) and (c) \(q_f \in \mathcal{E}\). Then by (b) and (d), \(q_f \in \Sigma_R\). Since also \(r_X \in \Sigma_R\) by (a), we have \(u_f \in \Sigma_R\), and \((R, r)\) is direct.

REMARKS 4.6. (1) Sufficient for the condition (b), \(q_f \in \mathcal{S}\), to hold in 4.5 and in 4.7 below is that \(\mathcal{S}\) be closed under pullback and each \(r_Y \in \mathcal{S}\).

(2) The example in 1.5 shows that the conditions of Proposition 4.5 do not force \((R, r)\) to be a reflector. (Take \(\mathcal{E} = \mathcal{S} = \text{Mor}(\text{Set})\) in 1.5.)

We recall that a class \(\mathcal{U}\) of \(X\)-morphisms is called coessential (see e.g., [5]) if for every \(u \in \mathcal{U}\) and every \(f\) such that \(uf \in \mathcal{U}\) it follows that \(f \in \mathcal{U}\).

PROPOSITION 4.7. Let \((R, r)\) be a reflector. Suppose that \(X\) has pullbacks of type \((\ast)\) and \(\mathcal{S}\) is a class of \(X\)-morphisms such that:

(a) For each \(X \in \text{Ob } X\), \(r_X \in \mathcal{S}\);
(b) For all pullbacks of type \((\ast)\), the morphism \(q_f \in \mathcal{S}\);
(c) \(\mathcal{S} \subseteq \mathcal{D}(R)\) is coessential.

Then \(R\) is direct.

Proof. In the diagram \((\ast)\), \(q_f \in \mathcal{S}\). Both \(q_f\) and \(r_X \in \mathcal{D}(R)\) so \(q_f u_f \in \mathcal{S} \cap \mathcal{D}(R)\) giving \(u_f \in \mathcal{S} \cap \mathcal{D}(R)\). But \(r_X \in \mathcal{M}(R)\) so \(u_f \in \mathcal{M}(R)\) and hence by Lemma 4.4 \(R\) is direct.

LEMMA 4.8. If \(R\) is a mono-reflection then \(\mathcal{D}(R) = \text{Epi } X\).
Proof. It is always true that $\text{Epi} X \subseteq D(\mathcal{R})$. If $f: X \to Y \in D(\mathcal{R})$ and there are $u, v: Y \to Z$ with $uf = vf$ then considering $r_Z$ one gets that $r_Zu = r_Zv$ and therefore $u = v$.

PROPOSITION 4.9. Let $X$ be a subcategory of $\text{Top}$ such that:

(a) Pullbacks in $X$ exist and are pullbacks in $\text{Top}$;
(b) $\text{Epi} X = \text{Mor} X \cap \{\text{Dense continuous maps}\}$.

Then every embedding-reflective subcategory $\mathcal{R}$ of $X$ is direct.

Proof. Let $\delta$ be the class of topological embeddings. By Lemma 4.8 $D(\mathcal{R}) = \text{Epi} X$. It is well known that embeddings are closed under pullback (so that condition (b) of 4.7 is satisfied) and that the class $\delta \cap D(\mathcal{R})$ (the dense embeddings) is coessential. Hence we conclude from Proposition 4.7 that the reflector to $\mathcal{R}$ is direct.

5. Examples

We shall use the following names for categories: $\text{Top}$ – topological spaces, $\text{Unif}$ – uniform spaces, $\text{Quu}$ – quasi-uniform spaces, $\text{Prox}$ – proximity spaces, $\text{BiTop}$ – bitopological spaces; a subscript 0 will indicate the corresponding full subcategory with $T_0$ topology.

5.1. APPLICATIONS OF PROPOSITION 4.5

The directness of the reflectors to the following subcategories $\mathcal{R}$ of $X$ can be concluded from Proposition 4.5.

<table>
<thead>
<tr>
<th>$X$</th>
<th>$\delta$</th>
<th>$\epsilon$</th>
<th>$\mathcal{R}$</th>
<th>$\Sigma_\mathcal{R}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Top}_0$</td>
<td>Embeddings</td>
<td>Epimorphisms</td>
<td>Sober spaces</td>
<td>$\delta \cap \epsilon$</td>
</tr>
<tr>
<td>$\text{Unif}_0$</td>
<td>Embeddings</td>
<td>Epimorphisms</td>
<td>Complete +$T_0$</td>
<td>$\delta \cap \epsilon$</td>
</tr>
<tr>
<td>$\text{Quu}_0$</td>
<td>Embeddings</td>
<td>Epimorphisms</td>
<td>Bicomplete +$T_0$</td>
<td>$\delta \cap \epsilon$</td>
</tr>
<tr>
<td>$\text{Top}$</td>
<td>Initial maps</td>
<td>$b$-dense maps</td>
<td>Sober spaces</td>
<td>$\delta \cap \epsilon$</td>
</tr>
<tr>
<td>$\text{Unif}$</td>
<td>Initial maps</td>
<td>Dense maps</td>
<td>Complete +$T_0$</td>
<td>$\delta \cap \epsilon$</td>
</tr>
<tr>
<td>$\text{Quu}$</td>
<td>Initial maps</td>
<td>Sup-dense</td>
<td>Bicomplete +$T_0$</td>
<td>$\delta \cap \epsilon$</td>
</tr>
</tbody>
</table>
5.2. \( T_0 \)-REFLECTION IN \( \text{TOP} \)

(Joint work with H.-P. A. Künzi.) We give a choice-free proof that the \( T_0 \) reflection \((R, r)\) in \( \text{Top} \) is direct. Given \( f: X \to Y \) in \( \text{Top} \), we form the diagram (*) and then let \( q^*: RP \to RX \) be the extension of \( q \) through \( r_P \). Since pullbacks preserve initial maps and \( r_Y \) is initial, so is \( q \). We note that any continuous map is initial iff the topology of its domain is the preimage of the topology of its codomain. To prove \( q^* \) initial, we consider any open set \( U \) in \( RP \). Since \( q \) is initial and \( r_P^{-1}U \) is open in \( P \), there is an open \( V \) in \( RX \) such that \( r_P^{-1}U = q^{-1}V \). Since \( r_P \) is surjective, \( q^*^{-1}V = U \), and thus \( q^* \) is initial. Since \( r_X = q^*r_Pu \) is surjective, so is \( q^* \), and having \( T_0 \) domain, \( q^* \) is an embedding. Thus \( q^* \) is a homeomorphism, \( q^* = (Ru)^{-1} \), and \( u \in \Sigma_R \). Thus the \( T_0 \)-reflection is direct.

5.3. \( T_0 \)-REFLECTION IN SATURATED TOPOLOGICAL CATEGORIES

(Joint work with H.-P. A. Künzi.) A topological category \( X \) over \( \text{Set} \) is called saturated if for every object \( X \) of \( X \) the \( T_0 \) reflection \( r_X: X \to RX \) is an initial map [18]. Examples are \( \text{Top}, \text{BiTop}, \text{Unif}, \text{Prox}, \text{Quu}, \text{Qprox} \). For every saturated topological category the \( T_0 \)-reflection is direct. Our proof coincides with the above proof in 5.2 except for the portion where \( q^* \) is shown to be initial. There we invoke the Axiom of Choice: Since \( r_P \) is surjective, it has a section \( s \), and indeed \( s: RP \to P \) is an \( X \)-morphism since \( r_P \) is initial. Consider any \( \text{Set} \)-morphism \( h: A \to RP \) such that \( q^*h: A \to RX \) is an \( X \)-morphism. Now \( q(sh) = q^*h \) and \( q \) being initial, \( sh: A \to P \) is an \( X \)-morphism. But \( h = r_P(sh) \) and so \( h: A \to RP \) is an \( X \)-morphism so that \( q^* \) is initial.

5.4. EMBEDDING REFLECTIONS IN \( \text{Tych} \)

Proposition 4.9 applies in particular to \( \text{Tych} \). Thus we easily see for instance that \( \text{HComp} \) and \( \text{RealComp} \) are directly reflective in \( \text{Tych} \). (The directness of the Čech–Stone reflection is also a well known consequence of [12].)

5.5. TOTALLY BOUNDED REFLECTION IN \( \text{UNIF} \)

(Joint work with H.-P. A. Künzi.) Let \( (P, p) \) be the totally bounded reflector in \( \text{Unif} \). Denote the covering uniformity of a uniform space \( X \) by \( \mu_X \) and the underlying set of \( X \) by \( |X| \). It is easy to see that every uniformly continuous \( f: X \to Y \) has a factorisation

\[
X \xrightarrow{u_f} (|X|, f^{-1}\mu_Y \vee \mu_{PX}) \xrightarrow{p_f} Y,
\]

where \( u_f = 1_{|X|} \) and \( p_f \) acts as \( f \). Noting that \( \Sigma_P \) consists of all uniformly continuous proximity isomorphisms, we see that \( u_f \in \Sigma_P \) and \( p_f \) is \( P \)-perfect, so that by Proposition 2.1 the totally bounded reflection is direct.
Borubaev [2] showed that, in $\text{Unif}$: $f: X \to Y$ is pullback of $Pf$ along $p_Y \iff \mu_X = f^{-1}\mu_Y \cup \mu_{PX}$, expressing the latter condition by:

(B) For each $\alpha \in \mu_X$ there exists $\beta \in \mu_Y$ and finite $\gamma \in \mu_X$ such that

$\alpha$ is refined by $f^{-1}\beta \cap \gamma$.

Thus $f$ is $P$-perfect iff $f \in \Sigma^1_P$, iff $f$ satisfies (B).

5.6. SAMUEL COMPACTIFICATION IN $\text{UNIF}_0$

(Joint work with H.-P. A. Künzi.) The Samuel compactification $(R, r)$ in $\text{Unif}_0$ is the composite of the totally bounded reflector $(P, p)$ followed by the completion $(K, k)$. Letting $\delta$ be the class of all uniformly continuous topological embeddings, we see by Lemma 4.8 that $\delta \cap \mathcal{D}(R)$ is the class of all uniformly continuous dense topological embeddings. Thus by Proposition 4.7 the Samuel compactification is a direct reflector.

Borubaev [2] defined $f: X \to Y$ in $\text{Unif}_0$ to be uniformly perfect if $f$ is a perfect map of the underlying topological spaces and satisfies condition (B) of 5.5. The relation to the Samuel compactification $(R, r)$ is the following:

$f \in \Sigma^1_R \iff f$ is $R$-perfect (i.e. pullback of $Rf$)

$\iff f$ is uniformly perfect (Borubaev)

$\iff f$ satisfies (B) and $(Rf)[RX - X] \subseteq RY - Y$

$\iff f$ satisfies (B) and for each Cauchy filter $\mathcal{F}$ in $X$,

if $f\mathcal{F}$ converges in $Y$ then $\mathcal{F}$ converges in $X$.

The following example shows that the condition $(Rf)[RX - X] \subseteq RY - Y$ does not imply that $f$ is uniformly perfect: Let $X = (\mathbb{N}, \text{discrete uniformity})$, $Y = (\mathbb{N}, \check{C}ech uniformity)$ and $f: X \to Y$ the identity function. Then $RX = RY = \beta\mathbb{N}$, and $Rf$ is the identity. However, $f$ is not a pullback of $Rf$ against $r_Y$, since then $X$ would be a uniform subspace of $Y \times \beta\mathbb{N}$ and therefore totally bounded, which is not the case.

Hager [9] already observed, for the Samuel compactification $(R, r)$, that the condition $(Rf)[RX - X] \subseteq RY - Y$ is equivalent to the underlying continuous mapping $Tf: TX \to TY$ being perfect in $\text{Tych}$ (here $T: \text{Unif}_0 \to \text{Tych}$ denotes the usual forgetful functor). It is striking that in the case of the Stone–Čech compactification in $\text{Tych}$ the ‘remainder mapped into remainder’ condition of [12] is equivalent to the mapping being a pullback of its extension, while this equivalence fails in the case of the Samuel compactification in $\text{Unif}_0$. The crux of the difference is that the Stone–Čech extension is a topological embedding in $\text{Tych}$ whereas the Samuel extension fails to be a uniform embedding. Several authors, notably [7, Example 9.3] and [9, Definition 2.1], have taken the weaker concept – that the underlying mapping in $\text{Tych}$ is perfect – as their notion of uniform perfectness.
5.7. TOPOLOGICAL GROUPS

In the category \( \text{TopGrp}_0 \) of Hausdorff topological groups, let \((R, r)\) denote the completion with respect to the two-sided (also called central) uniformity. This completion is known to be an epireflection \([16]\). Moreover, \( \Sigma_R \) is precisely the class of dense embeddings of topological groups \([5]\). From Proposition 4.5, taking \( \mathcal{E} \) the class of dense morphisms and \( \mathcal{D} \) the class of embeddings, we see that \((R, r)\) is direct.

5.8. NON-DIRECT EPIREFLECTION

The 0-dimensional \( T_0 \) spaces give an example of an epireflective subcategory of \( \text{Top} \) that is not directly reflective.

Let \( Y = \mathbb{N} \cup \{\infty_1, \infty_2\} \) (the natural numbers with two points at infinity) and let \( X \) be a singleton space with inclusion map \( f: X \to Y \) mapping \( X \) onto \( \infty_1 \).

Let the topology on \( Y \) be such that \( \mathbb{N} \) is discrete as a subspace and the basic neighbourhoods of \( \infty_i \) are those of the form \( \{m \mid m \geq n\} \cup \{\infty_i\} \) for \( n \in \mathbb{N} \).

If \((R, r)\) is the 0-dimensional \( T_0 \) reflection, then we can form the usual pullback \((*)\) and it is routine to check that \( RX \) is not isomorphic to \( RP \).

5.9. TOPOLOGICAL COMPLETIONS

Let \( T: \text{Unif}_0 \to \text{Tych} \) denote the forgetful functor from Hausdorff uniform spaces to Tychonoff spaces, and let \( F: \text{Tych} \to \text{Unif}_0 \) be any section of \( T \). Let \((K, k)\) be the completion in \( \text{Unif}_0 \). We form the pointed endofunctor \((TKF, TkF)\) in \( \text{Tych} \).

It is clear that \((TKF, TkF)\) is a prereflection \([3]\); it will thus be a reflection iff it is idempotent. We have:

\[(TKF, TkF) \text{ is a reflection } \iff (TKF, TkF) \text{ is direct.}\]

For, if \((TKF, TkF)\) is direct, the presence of a terminal object in \( \text{Tych} \) ensures by Proposition 1.3 that we have a reflection. Conversely, let \((TKF, TkF)\) be a reflection. Taking \( X = \text{Tych} \) in Proposition 4.9 we see that \((TKF, TkF)\) is direct.

Reflective topological completions of the type \((TKF, TkF)\) exist in profusion: the corresponding embedding-reflective subcategories of \( \text{Tych} \) range from the compact spaces (when \( F \) is the Čech functor \( C^* \)) to the topologically complete spaces (when \( F \) is the fine uniformity functor) \([3]\). An example is known of a \( T \)-section \( F \) for which \((TKF, TkF)\) is non-idempotent \([6]\).

5.10. TOPOLOGICAL BICOMPLETIONS

Let \( T: \text{Quu}_0 \to \text{Top}_0 \) denote the usual ‘first topology’ forgetful functor from the category of \( T_0 \) quasi-uniform spaces to the category of \( T_0 \) topological spaces.
Let $F : \text{Top}_0 \to \text{Quu}_0$ be any section of $T$. Let $(K, k)$ be the bicompletion in $\text{Quu}_0$—known to be an embedding-reflection. The $T$-section $F$ is called upper $K$-true if $KFX$ is finer than $FTKFX$ for each $X$ in $\text{Top}_0$. We consider the pointed endofunctor $(R, r) := (TKF, TkF)$ in $\text{Top}_0$. It is known [3] that:

$F$ is upper $K$-true $\iff$ $(R, r)$ is well-pointed $\iff$ $(R, r)$ is a prereflection.

We now show:

$(R, r)$ is direct and $F$ is upper $K$-true $\iff$ $(R, r)$ is a reflection.

Indeed, ‘$\Rightarrow$’ follows from the above result by Proposition 1.3, since $\text{Top}_0$ has a terminal object. Conversely, let $(R, r)$ be a reflection. As above, $F$ is upper $K$-true. Let $R = \text{Fix}(R, r)$. Since $(R, r)$ is pointwise an embedding, Lemma 4.8 shows that $\mathcal{D}(R) = \text{Epi } \text{Top}_0$, which is the class of $b$-dense mappings. Let $\mathcal{D}$ be the class of embeddings in $\text{Top}_0$. The class $\delta \cap \mathcal{D}(R)$ of $b$-dense embeddings is coessential [5] (one can see this by applying the $b$-topology functor to this class and using the fact that the dense embeddings form a coessential class in $\text{Top}$). It follows by Proposition 4.6 that $(R, r)$ is direct.

The above result is pursued further in [4]. It is still an open problem to characterise those $T$-sections $F$ for which $(TKF, TkF)$ is direct.

Acknowledgements

The first author acknowledges grants from the Foundation for Research Development and the University of Cape Town to the Topology and Category Theory Group. He also acknowledges support from the second author’s grant at the University of L’Aquila during several visits, as well as from the grant of Hans-Peter Künzi at the University of Berne during a collaboration that led to the Examples 5.2, 5.3, 5.5, 5.6.

The second author acknowledges grants from the Italian MURST and support from the first author’s grant during visits to the University of Cape Town. Assistance from NATO Collaborative Research Grant (No. 940847) is also acknowledged.

The third author acknowledges support from the grants of both the other authors as well as the Universities of Cape Town and L’Aquila and a postdoctoral scholarship from the Foundation for Research Development.

References


JURASSIC AGE

Linosaurus realis H.H.
real longliner

Compactificatio unipunctata Alexandroff